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Tail of the stationary solution of the stochastic equation $Y_{n+1} = a_n Y_n + b_n$ with Markovian coefficients

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Abstract

In this paper, we deal with the real stochastic difference equation $Y_{n+1} = a_n Y_n + b_n$, $n \in \mathbb{Z}$, where the sequence (a_n) is a finite state space Markov chain. By means of the renewal theory, we give a precise description of the situation where the tail of its stationary solution exhibits power law behavior.

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1. Introduction

We study the following stochastic difference equation:

$$Y_{n+1} = a_n Y_n + b_n, \quad n \in \mathbb{Z}, \quad (1)$$

where (a_n) is a real, finite state space Markov chain, and (b_n) is a sequence of real i.i.d. random variables. Equations of type (1) have many applications in stochastic modeling and statistics. Most of previously studied cases deal with i.i.d.

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multiplicative coefficients (a_n) : see [13,14,16,9]. For more recent work, see also [15]. Here, we study the Markovian case. In statistical literature, Eq. (1) defines a so-called Markov-switching auto-regression. See [11] for interesting applications in econometrics. Such stochastic recursions are also a basic tool in queuing theory: see [3].

We assume throughout this paper that the following conditions are satisfied:

$$\begin{aligned} \mathbb{E} \log |a_0| &< 0, \\ \mathbb{E} \log^+ |b_0| &< \infty. \end{aligned} \quad (2)$$

If in addition (a_n, b_n) is stationary and ergodic, Brandt [5] proved that Eq. (1) has a unique stationary solution (Y_n) , where

$$Y_n = \sum_{k=0}^{\infty} a_{n-1} a_{n-2} \cdots a_{n-k} b_{n-1-k}, \quad n \in \mathbb{Z}.$$

In the following, (Y_n) will always denote the stationary solution of Eq. (1). We deal with the tail of Y_1 : we investigate the asymptotic behavior of $\mathbb{P}(xY_1 > t)$, when t tends to infinity, and where $x \in \{-1, 1\}$. Our approach is based on renewal-theoretic methods as developed in [16,9].

Our main results are the following two theorems, depending on the a_n being positive or not. Let \mathbb{R} be the set of real numbers, and \mathbb{R}_+^* the set of positive real numbers.

Theorem 1. *Let (a_n) be an irreducible, aperiodic, stationary Markov chain, with state space $E = \{e_1, \dots, e_p\} \subset \mathbb{R}_+^*$, transition matrix $P = (p_{ij})$ and stationary law ν . Let (b_n) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence (a_n) . If the following conditions are satisfied:*

- (1) *there is a $\lambda > 0$ so that the matrix $P_\lambda = \text{diag}(e_i^\lambda)P'$ has spectral radius 1 (P' denotes the transpose of P),*
- (2) *the numbers $\log e_i$ are not integral multiples of the same number,*
- (3) *there is a $\delta > 0$ such that $\mathbb{E}|b_0|^{\lambda+\delta} < \infty$,*

then we have for $x \in \{-1, 1\}$

$$t^\lambda \mathbb{P}(xY_1 > t) \xrightarrow[t \rightarrow \infty]{} L(x),$$

where $L(1) + L(-1)$ is positive. If $b_0 \geq 0$ a.s., then $L(-1) = 0$, and $L(1) > 0$. If $b_0 \leq 0$ a.s., then $L(1) = 0$, and $L(-1) > 0$.

Theorem 2. *Let (a_n) be an irreducible, aperiodic, stationary Markov chain, with state space $E = \{e_1, \dots, e_p\} \subset \mathbb{R}$ such that e_1, \dots, e_ℓ are positive and $e_{\ell+1}, \dots, e_p$ are negative for a $0 \leq \ell \leq p-1$ ($\ell = 0$ means that all the e_i are negative). Let $P = (p_{ij})$ be its transition matrix and ν its stationary law. Let (b_n) be a sequence of non-zero real i.i.d. random variables, and independent of the sequence (a_n) . If the following conditions*

are satisfied:

- (1) there is a $\lambda > 0$ so that $P_\lambda = \text{diag}(|e_i|^\lambda)P'$ has spectral radius 1,
- (2) the numbers $\log |e_i|$ are not integral multiples of the same number,
- (3) there is a $\delta > 0$ such that $\mathbb{E}|b_0|^{\lambda+\delta} < \infty$,

then we have, for $x \in \{-1, 1\}$,

$$t^\lambda \mathbb{P}(xY_1 > t) \xrightarrow[t \rightarrow \infty]{} L(x),$$

where $L(1) + L(-1)$ is positive. If in addition P' is ℓ -irreducible (see Definition 3) then $L(1) = L(-1) > 0$.

The last two hypotheses in these theorems are the same as in the i.i.d. case. In particular, Hypothesis (2) ascertains that the distribution of Y_1 is non-lattice, and it is equivalent to requiring that the subgroup generated by the $\log e_i$ be dense in \mathbb{R} . On the contrary, Assumption (1) comes from the Markovian dependence considered here. Indeed, we will prove in Section 4.1 that the spectral radius $\rho(P_\lambda)$ of the matrix P_λ can be computed from the formula $\rho(P_\lambda) = \lim(\mathbb{E}|a_0, \dots, a_{1-n}|^\lambda)^{1/n}$. Therefore, this assumption is a suitable substitute for the classical relation $\mathbb{E}|a_0|^\lambda = 1$ assumed in the i.i.d. case.

Note that the assumption of independence between the two sequences (a_n) and (b_n) can be avoided. Let \mathcal{F}_{-n} be the σ -field generated by a_0, \dots, a_{-n} and b_0, \dots, b_{-n} . Then (b_n) is only required to be a sequence of random variables such that (a_n, b_n) is a stationary process, and $b_{-(n+1)}$ is independent of \mathcal{F}_{-n} . We also need one more assumption (also assumed in the i.i.d. case): for all i , $\mathbb{P}(b_0 + a_0x = x | a_0 = e_i) < 1$. The proofs run exactly the same, except that of Lemma 3, where $\min_{1 \leq i \leq p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)$ must be replaced by $\min_{1 \leq i \leq p} \mathbb{P}(|b_0 + (a_0 - 1)m_0| > \varepsilon | a_0 = e_i)$. And thanks to the new assumption, we can again choose a positive ε such that the latter minimum is positive.

As the mapping $\lambda \mapsto \log \rho(P_\lambda)$ is convex (see Section 4.1), that its right-hand derivative at 0 is negative and $\rho(P_0) = \rho(P) = 1$, only two cases may occur. Either for all $\lambda > 0$, $\rho(P_\lambda) < 1$, in which case we can prove that $\mathbb{E}|Y_1|^\lambda < \infty$ for all λ , provided $\mathbb{E}|b_0|^\lambda < \infty$ (see Proposition 3), and therefore $\mathbb{P}(|Y_1| > t) = o(t^{-\lambda})$ for all λ ; or there is a unique $\lambda > 0$ so that $\rho(P_\lambda) = 1$, this is the case we study here.

Similar results have already been proved in the i.i.d. multidimensional case: a_n are matrices and Y_n and b_n vectors. Renewal theory is used by Kesten [13] when the a_n either have a density or are non-negative. These results were extended by Le Page [16] to all i.i.d. random matrices satisfying similar assumptions as in our theorems. Finally Goldie [9] proved a new specific implicit renewal theorem and derived the same results as Kesten in the i.i.d. one-dimensional case. He also studies the tails of the stationary solutions of several other one-dimensional random equations with i.i.d. coefficients.

The paper is organized as follows. In Section 2, we introduce some notation and state a new renewal theorem. In Section 3 we derive the renewal equations corresponding to our problem. In Sections 4 and 5, we prove Theorem 1, Section 5 being dedicated to the proof that the sum of the limits is non-zero. And finally in Section 6 we prove Theorem 2.

2. A renewal theorem

Our approach is based on a new renewal theorem for systems of renewal equations. First, we give some notation and conventions that will apply throughout.

Let $F = (F_{ij})_{1 \leq i, j \leq p}$ be a matrix of distributions: non-decreasing, right-continuous functions from \mathbb{R} to \mathbb{R}_+ with limit 0 at $-\infty$.

Definition 1. For all $r \geq 1$ and all $p \times r$ vector or matrix H of Borel measurable, real valued functions H_{ij} on \mathbb{R} that are bounded on compact intervals, we define the convolution product $F * H$ by

$$(F * H)_{ij}(t) = \sum_{k=1}^p \int_{-\infty}^{\infty} H_{kj}(t-u) F_{ik}(du),$$

where it exists.

We study the renewal equation $Z = F * Z + G$, where $G = (G_1, \dots, G_p)'$ is a vector of Borel measurable, real valued functions, bounded on compact intervals, and $Z = (Z_1, \dots, Z_p)'$ is a vector of functions. The renewal theorem will give the limit of Z at $+\infty$.

For all real t , set:

- $B = (b_{ij})_{1 \leq i, j \leq p}$ where $b_{ij} = \int u F_{ij}(du)$ if it exists, the expectation of F ,
- $F^{(0)}(t) = (\delta_{ij}(t))_{1 \leq i, j \leq p}$ where $\delta_{ij}(t) = \mathbf{1}_{t \geq 0}$ if $i = j$ and 0 otherwise, so that $F^{(0)} * H = H$ for all H as in the definition above,
- $F^{(n)}(t) = F * F^{(n-1)}(t)$, the n -fold convolution of F ,
- $U(t) = \sum_{n=0}^{\infty} F^{(n)}(t)$, the *renewal function* associated with F .

Assume that all the measures F_{ij} are finite:

$$F_{ij}(\infty) = \lim_{t \rightarrow \infty} F_{ij}(t) < \infty,$$

and that $F(\infty)$ is an irreducible matrix (see e.g. [12] for a definition and Perron–Frobenius theory). By Perron–Frobenius theorem, the spectral radius $\rho(F(\infty))$ of $F(\infty)$ is a simple eigenvalue with right and left positive eigenvectors. Assume that $\rho(F(\infty)) = 1$, and let m and u be two positive eigenvectors such that:

$$F(\infty)m = m, \quad u'F(\infty) = u', \quad \sum_{i=1}^p m_i = 1, \quad \sum_{i=1}^p u_i m_i = 1.$$

Assume also that the sequence $(\|F(\infty)^n\|)$ is bounded (for instance if $F(\infty)$ is aperiodic, this is true). We recall the following definition:

Definition 2. The matrix of distributions F is *lattice* if the following conditions are satisfied:

- For all $i \neq j$, F_{ij} is concentrated on a set of the form $b_{ij} + \lambda_{ij}\mathbb{Z}$.
- For all i , F_{ii} is concentrated on a set of the form $\lambda_{ii}\mathbb{Z}$.

- Each λ_{ij} is an integral multiple of the same number.
We take λ to be the largest such number.
- For all a_{ij} , a_{jk} , a_{ik} points of increase of F_{ij} , F_{jk} , F_{ik} , respectively, $a_{ij} + a_{jk} - a_{ik}$ is an integral multiple of λ .

Our basic tool is the following renewal theorem from [17]. It extends a previous result of Crump [7] and Athreya and Rama Murthy [4] which deals with the case where each distribution F_{ij} has support on \mathbb{R}_+ .

Renewal Theorem A. *Assume that F is a matrix of distributions satisfying the assumptions above, that it is non-lattice, and that*

- (1) *its expectation B exists,*
- (2) *for all $t \in \mathbb{R}$, $U(t)$ is finite.*

*If in addition G is directly Riemann integrable (see [8]), and $Z = U * G$ exists, then for all i , we have:*

$$\lim_{t \rightarrow \infty} Z_i(t) = cm_i \sum_{j=1}^p \left[u_j \int_{-\infty}^{\infty} G_j(y) dy \right],$$

where m and u are the eigenvectors defined above and $c = (u' B m)^{-1}$ (under these assumptions, $u' B m \neq 0$).

We also recall Theorem 2.3 of [4] that will be used in Section 5.

Renewal Theorem B. *Let F be a non-lattice matrix of distributions with support on the positive half-line, such that*

- (1) $\rho(F(0)) < 1$,
- (2) $F(\infty)$ *is finite, irreducible and aperiodic.*

Assume also that there is a $\alpha > 0$ such that $\rho(F_\alpha) = 1$, where $(F_\alpha)_{ij} = \int_0^\infty e^{-\alpha u} F_{ij}(du)$. Then for all $h > 0$, and all i, j , we have

$$\lim_{t \rightarrow \infty} \int_t^{t+h} e^{-\alpha y} U_{ij}(dy) = cm_i u_j h,$$

where m and u are right and left eigenvectors of F_α , with the same normalization as above, $c = (u' B m)^{-1}$, and $B = (b_{ij})$ with $b_{ij} = \int_0^\infty u e^{-\alpha u} F_{ij}(du)$, c being interpreted as zero if some b_{ij} is equal to infinity.

Note that this theorem can now be seen as a corollary of Theorem A. Indeed, the first assumption ascertains that $U(t)$ is finite for all t . In the positive case, the expectation B and the convolution product $U * G$ are always defined (possibly infinite). Applying Theorem A with $F = F_\alpha$ and $G = \mathbf{1}_{[t, t+h]}$ (which is obviously directly Riemann integrable) yields Theorem B.

3. The renewal equations

Let

$$z(x, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(xY_1 > u) du.$$

For all $(x, t) \in \{-1, 1\} \times \mathbb{R}$, we have: $z(x, t) = \sum_{i=1}^p Z_i(x, t)$, where

$$Z_i(x, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(xY_1 > u, a_0 = e_i) du.$$

Besides, $Y_1 = a_0 Y_0 + b_0$, thus for all $(x, u) \in \{-1, 1\} \times \mathbb{R}$, and for all i we have

$$\mathbb{P}(xY_1 > u, a_0 = e_i) = \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i) + \psi_i(x, u),$$

where

$$\psi_i(x, t) = \mathbb{P}(t - xb_0 < xa_0 Y_0 \leq t, a_0 = e_i) - \mathbb{P}(t < xa_0 Y_0 \leq t - xb_0, a_0 = e_i).$$

Let $G_i(x, t) = e^{-t} \int_0^{e^t} u^\lambda \psi_i(x, u) du$. We get

$$z(x, t) = \sum_{i=1}^p \left[e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i) du + G_i(x, t) \right].$$

Now we need to distinguish two cases. Indeed, we make a change of variable that involves the sign of a_0 . We start with the easier special case when all the states of our Markov chain are positive, therefore the sign of a_0 is non-random.

3.1. Positive case

Suppose all the states of our Markov chain are positive. Then for all (x, t) in $\{-1, 1\} \times \mathbb{R}$, and all i , we have, thanks to a simple change of variable,

$$e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i) du = e^{-(t - \log e_i)} e_i^\lambda \int_0^{e^{t - \log e_i}} u^\lambda \mathbb{P}(xY_0 > u, a_0 = e_i) du. \quad (3)$$

The Markov property and the stationarity of (Y_n) yield

$$\begin{aligned} \mathbb{P}(xY_0 > u, a_0 = e_i) &= \sum_{j=1}^p \mathbb{P}(xY_0 > u, a_0 = e_i, a_{-1} = e_j) \\ &= \sum_{j=1}^p \mathbb{P}(xY_0 > u | a_{-1} = e_j) v(e_j) p_{ji} \\ &= \sum_{j=1}^p \mathbb{P}(xY_1 > u | a_0 = e_j) v(e_j) p_{ji}. \end{aligned}$$

Thus, we get the following formula for Z_i :

$$\begin{aligned} Z_i(x, t) &= \sum_{j=1}^p \left[e^{-(t-\log e_i)} e_i^\lambda \int_0^{e^{t-\log e_i}} u^\lambda \mathbb{P}(xY_1 > u, a_0 = e_j) p_{ji} du \right] + G_i(x, t) \\ &= e_i^\lambda \sum_{j=1}^p [p_{ji} Z_j(x, t - \log e_i)] + G_i(x, t). \end{aligned}$$

We can rewrite this system of equations as follows:

$$\forall 1 \leq i \leq p, \quad Z_i(x, t) = \sum_{j=1}^p F_{ij} * Z_j(x, t) + G_i(x, t),$$

where $F_{ij}(t) = e_i^\lambda p_{ji} \mathbf{1}_{t \geq \log e_i}$ are distribution functions. Let $Z = (Z_1, \dots, Z_p)'$, $G = (G_1, \dots, G_p)'$ and F be the matrix $F = (F_{ij})$. With the notations of Section 2 we have the following system of renewal equations for fixed x :

$$Z(x, t) = F * Z(x, t) + G(x, t). \quad (4)$$

3.2. General case

Now we study the general case. In order to determine the sign of a_0 , we classify our states according to their sign: assume there is a $0 \leq \ell \leq p-1$ so that $e_1, \dots, e_\ell > 0$ and $e_{\ell+1}, \dots, e_p < 0$. Then Eq. (3) becomes

$$\begin{aligned} e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(xa_0 Y_0 > u, a_0 = e_i) du \\ = e^{-(t-\log |e_i|)} |e_i|^\lambda \int_0^{e^{t-\log |e_i|}} u^\lambda \mathbb{P}(x \cdot e_i Y_0 > u, a_0 = e_i) du, \end{aligned}$$

where $x \cdot e_i$ denotes the sign of $x e_i$. To get similar equations as in the positive case, we introduce $2p$ new functions:

$$\begin{aligned} \forall 1 \leq i \leq p, \quad Z_i^+(t) &= Z_i(1, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(Y_1 > u, a_0 = e_i) du, \\ \forall 1 \leq i \leq p, \quad Z_i^-(t) &= Z_i(-1, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(-Y_1 > u, a_0 = e_i) du. \end{aligned}$$

Following the same steps as in the positive case, we get

$$\begin{aligned} \forall 1 \leq i \leq \ell, \quad Z_i^+(t) &= |e_i|^\lambda \sum_{j=1}^p p_{ji} Z_j^+(t - \log |e_i|) + G_i(1, t), \\ \forall \ell+1 \leq i \leq p, \quad Z_i^+(t) &= |e_i|^\lambda \sum_{j=1}^p p_{ji} Z_j^-(t - \log |e_i|) + G_i(1, t), \end{aligned}$$

$$\begin{aligned} \forall 1 \leq i \leq l, \quad Z_i^-(t) &= |e_i|^\lambda \sum_{j=1}^p p_{ji} Z_j^-(t - \log |e_i|) + G_i(-1, t), \\ \forall l+1 \leq i \leq p, \quad Z_i^-(t) &= |e_i|^\lambda \sum_{j=1}^p p_{ji} Z_j^+(t - \log |e_i|) + G_i(-1, t), \end{aligned} \quad (5)$$

that we can also rewrite as a system of renewal equations: set

$$\tilde{Z} = (Z_1^+, \dots, Z_p^+, Z_1^-, \dots, Z_p^-)' \quad \text{and} \quad \tilde{G} = (G_1^+, \dots, G_p^+, G_1^-, \dots, G_p^-)',$$

where $G_i^+(t) = G_i(1, t)$ and $G_i^-(t) = G_i(-1, t)$. Define the $2p \times 2p$ matrix $\tilde{F} = (\tilde{F}_{ij})$ by:

$$\begin{aligned} \tilde{F}_{ij}(t) &= |e_i|^\lambda p_{ji} \mathbf{1}_{t \geq \log |e_i|} && \text{if } 1 \leq i \leq l \text{ and } 1 \leq j \leq p, \\ & && \text{or } p+l+1 \leq i \leq 2p \text{ and } 1 \leq j \leq p, \\ & && \text{or } l+1 \leq i \leq p+l \text{ and } p+1 \leq j \leq 2p, \\ \tilde{F}_{ij}(t) &= 0 && \text{otherwise,} \end{aligned}$$

where $\tilde{i} = i \bmod p$ (see Eq. (19) for an explicit matrix form of \tilde{F}). Now Eq. (5) becomes

$$\tilde{Z}(t) = \tilde{F} * \tilde{Z}(t) + \tilde{G}(t).$$

4. Part I of the proof of Theorem 1

Throughout this section, we assume that the hypotheses of Theorem 1 are satisfied. In order to apply Renewal Theorem A, we have to check that F and G satisfy its hypotheses. Note first that $F_{ij}(\infty) = e_i^\lambda p_{ji} < \infty$ and that B the expectation of F is well defined. Indeed, $b_{ij} = e_i^\lambda p_{ji} \log e_i < \infty$. The assumption that the $\log e_i$ are not integral multiples of the same number implies that F is non-lattice. The other points are proved in the following sections.

4.1. Finiteness of U

Remember that $U = \sum_{k=0}^{\infty} F^{(k)}$. We have to check that $U(t) < \infty$ for all real t . First, we study the spectral radius of the matrices $P_\alpha = \text{diag}(e_i^\alpha) P'$, i.e. $(P_\alpha)_{ij} = e_i^\alpha p_{ji}$, for $\alpha > 0$.

Proposition 1. *For all $\alpha > 0$, we have*

$$\rho(P_\alpha) = \lim_k (\mathbb{E}|a_0, \dots, a_{-k}|^\alpha)^{1/k}.$$

Proof. We have

$$\mathbb{E}|a_0 a_{-1}, \dots, a_{-k}|^\alpha = \sum_{i_1, \dots, i_{k+1}} \mathbb{P}(a_0 = e_{i_1}, \dots, a_{-k} = e_{i_{k+1}}) |e_{i_1}, \dots, e_{i_{k+1}}|^\alpha$$

$$\begin{aligned}
 &= \sum_{i_1, \dots, i_{k+1}} p_{i_2 i_1} \dots p_{i_{k+1} i_k} v(e_{i_{k+1}}) |e_{i_1}, \dots, e_{i_{k+1}}|^\alpha \\
 &= \sum_{i,j} (P_\alpha^k)_{ij} v(e_j) e_j^\alpha,
 \end{aligned}$$

where P_α^k is the k th power of the matrix P_α . Rewrite this equation as

$$\mathbb{E}|a_0, a_{-1}, \dots, a_{-k}|^\alpha = \mathbf{1} P_\alpha^k D_\alpha, \quad (6)$$

where $\mathbf{1}$ denotes the constant row vector with all coordinates equal to 1, and D_α is the column vector with coordinates $v(e_j) e_j^\alpha$. As P , and thus P_α , is aperiodic, Theorem 8.5.1 of [12] yields

$$\frac{P_\alpha^k}{\rho^k(P_\alpha)} \xrightarrow{k \rightarrow \infty} A_\alpha \quad (7)$$

where A_α is a constant positive matrix. Thus $(\mathbf{1} P_\alpha^k D_\alpha)^{1/k} \xrightarrow{k \rightarrow \infty} \rho(P_\alpha)$. \square

The following corollary is obvious.

Corollary 1. *The mapping $\alpha \mapsto \log(\rho(P_\alpha))$ is convex on \mathbb{R}_+ .*

Proposition 2. *The right-hand derivative of $\alpha \mapsto \log(\rho(P_\alpha))$ at zero is negative.*

To prove this proposition, we need another expression for $\rho(P_\alpha)$. We set $\mathbb{E}_e[\cdot] = \mathbb{E}[\cdot | a_0 = e]$ for all $e \in E$.

Lemma 1. *Set $h_n(\alpha) = \max_{e \in E} \mathbb{E}_e[(a_{-1}, \dots, a_{-n})^\alpha]$. Then we have $\rho(P_\alpha) = \inf_n (h_n(\alpha))^{1/n}$.*

Proof. We first prove that the sequence (h_n) is sub-multiplicative. Indeed, set $e \in E$. We have

$$\begin{aligned}
 \mathbb{E}_e[(a_{-1}, \dots, a_{-n} a_{-n-1}, \dots, a_{-n-m})^\alpha] &= \mathbb{E}_e[(a_{-1}, \dots, a_{-n})^\alpha \mathbb{E}_{a_{-n}}[(a_{-1}, \dots, a_{-m})^\alpha]] \\
 &\leq h_m(\alpha) \mathbb{E}_e[(a_{-1}, \dots, a_{-n})^\alpha] \\
 &\leq h_m(\alpha) h_n(\alpha),
 \end{aligned}$$

as $\mathbb{E}_{a_{-n}}[(a_{-1}, \dots, a_{-m})^\alpha] \leq h_m(\alpha)$. Thus $\lim_n (h_n(\alpha))^{1/n} = \inf_n (h_n(\alpha))^{1/n}$. Besides, we have

$$\begin{aligned}
 \mathbb{E}|a_0 a_{-1}, \dots, a_{-n}|^\alpha &= \sum_{e \in E} \mathbb{E}_e|a_{-1}, \dots, a_{-n}|^\alpha e^\alpha v(e) \\
 &\leq h_n(\alpha) \sum_{e \in E} e^\alpha v(e).
 \end{aligned}$$

As $\sum_{e \in E} e^\alpha v(e) > 0$, Proposition 1 yields

$$\rho(P_\alpha) \leq \lim_n (h_n(\alpha))^{1/n}.$$

On the other hand, set e_n such that $h_n(\alpha) = \mathbb{E}_{e_n}[(a_{-1}, \dots, a_{-n})^\alpha]$. The equation above then yields

$$\begin{aligned}
 \mathbb{E}|a_0 a_{-1}, \dots, a_{-n}|^\alpha &\geq h_n(\alpha) e_n^\alpha v(e_n) \\
 &\geq C h_n(\alpha),
 \end{aligned}$$

where $C = \min_{e \in E} e^{\alpha} v(e) > 0$. Hence we also have

$$\rho(P_{\alpha}) \geq \lim_n (h_n(\alpha))^{1/n}.$$

As $\lim_n (h_n(\alpha))^{1/n} = \inf_n (h_n(\alpha))^{1/n}$, the lemma is proved. \square

Proof of Proposition 2. For any fixed n , set $e_n \in E$ such that $h_n(\alpha) = \mathbb{E}_{e_n}[(a_{-1}, \dots, a_{-n})^{\alpha}]$. As the product a_{-1}, \dots, a_{-n} is bounded for a fixed n , we have

$$\frac{\partial}{\partial \alpha} h_n(\alpha) = \mathbb{E}_{e_n}[(a_{-1}, \dots, a_{-n})^{\alpha} \log(a_{-1}, \dots, a_{-n})],$$

hence

$$\left. \frac{\partial}{\partial \alpha} \right|_{\alpha=0} \frac{1}{n} \log h_n(\alpha) = \frac{1}{n} \mathbb{E}_{e_n}[\log(a_{-1}, \dots, a_{-n})].$$

For all $e \in E$, the Ergodic Theorem for stationary Markov chains yields

$$\frac{1}{n} \mathbb{E}_e[\log(a_{-1}, \dots, a_{-n})] \xrightarrow{n \rightarrow \infty} \mathbb{E} \log a_0 = \gamma < 0. \quad (8)$$

As the state space E is finite, this convergence is also uniform on E . Thus, for any sequence (e_n) in E we have

$$\frac{1}{n} \mathbb{E}_{e_n}[\log(a_{-1}, \dots, a_{-n})] \xrightarrow{n \rightarrow \infty} \gamma < 0.$$

Hence, there is an integer N such that

$$\left. \frac{\partial}{\partial \alpha} \right|_{\alpha=0} \frac{1}{N} \log h_N(\alpha) \leq \frac{\gamma}{2} < 0.$$

In particular, the mapping $\alpha \mapsto \frac{1}{N} \log h_N(\alpha)$ is negative on an interval of the form $]0, \varepsilon[$, with $\varepsilon > 0$. The preceding lemma then yields

$$\begin{aligned} \log \rho(P_{\alpha}) &= \inf_n \frac{1}{n} \log h_n(\alpha) \\ &\leq \frac{1}{N} \log h_N(\alpha), \end{aligned}$$

which is negative for all $\alpha \in]0, \varepsilon[$. But the mapping $\alpha \mapsto \log \rho(P_{\alpha})$ is convex and continuous on \mathbb{R}_+ , and takes the value 0 at 0. The result above implies that its right-hand derivative at 0 is negative (possibly $-\infty$). \square

We have $\rho(P_0) = 1$, and in addition, in the case we study here, $\rho(P_{\lambda}) = 1$, thus Proposition 2 and Corollary 1 easily yield the following corollary:

Corollary 2. For all $0 < \alpha < \lambda$, we have $\rho(P_{\alpha}) < 1$.

Now we can study U . By definition, $F(\infty) = P_{\lambda}$ is irreducible as P is and all e_i are non-zero. We have chosen λ so that $\rho(P_{\lambda}) = \rho(F(\infty)) = 1$. For all $\alpha \in]0, \lambda[$, we have $P_{\lambda-\alpha} = (e_i^{\lambda-\alpha} p_{ji}) = (\int e^{-\alpha u} F_{ij}(du))$. Corollary 2 yields $\rho(P_{\lambda-\alpha}) < 1$, so that the series $\sum_{n=0}^{\infty} (P_{\lambda-\alpha}^n)_{ij}$ is convergent for all i, j . As for all n , $(P_{\lambda-\alpha}^n)_{ij} = \int e^{-\alpha u} F_{ij}^{(n)}(du)$ holds,

then we have

$$(P_{\lambda-\alpha}^n)_{ij} \geq \int_{-\infty}^t e^{-\alpha u} F_{ij}^{(n)}(du) \geq e^{-\alpha t} \int_{-\infty}^t F_{ij}^{(n)}(du) = e^{-\alpha t} F_{ij}^{(n)}(t).$$

Thus, for all i, j and t , we have $U_{ij}(t) = \sum F_{ij}^{(n)}(t) \leq e^{\alpha t} \sum (P_{\lambda-\alpha}^n)_{ij} < \infty$.

4.2. Proof of $Z = U * G$

Iterating Eq. (4) yields:

$$Z = \sum_{k=0}^{n-1} [F^{(k)} * G] + F^{(n)} * Z.$$

It is thus sufficient to prove that $F^{(n)} * Z \rightarrow 0$. As seen in Section 3 we have

$$\begin{aligned} (F * Z)_i(x, t) &= \sum_{j=1}^p \left[e^{-(t-\log e_j)} \int_0^{e^{t-\log e_j}} e_i^\lambda p_{ji} u^\lambda \mathbb{P}(x Y_1 > u, a_0 = e_j) du \right] \\ &= e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(x a_0 Y_0 > u, a_0 = e_i) du. \end{aligned}$$

Similarly, we get for all n

$$(F^{(n)} * Z)_i(x, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(x a_0, \dots, a_{1-n} Y_{1-n} > u, a_0 = e_i) du.$$

And thus we have

$$\sum_{i=1}^p (F^{(n)} * Z)_i(x, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(x a_0, \dots, a_{1-n} Y_{1-n} > u) du.$$

But $a_0, \dots, a_{1-n} = \exp(\sum_{k=1}^n \log a_{1-k})$, thus Eq. (8) and Assumption (2) yield $a_0, \dots, a_{1-n} \rightarrow 0$. Hence for all $u > 0$, the bounded convergence theorem yields:

$$\mathbb{P}(x a_{1-n}, \dots, a_0 Y_{1-n} > u) \xrightarrow{n \rightarrow \infty} 0,$$

because $Y < \infty$ a.s. and is stationary. Thus $\sum_{i=1}^p (F^{(n)} * Z)_i(x, t) \rightarrow 0$ holds a.s. As all the terms in the sum are non-negative, each one tends to zero and we have $Z = U * G$ as required.

4.3. G is directly Riemann integrable

We first consider the moments of Y_1 .

Proposition 3. For all $0 \leq s < \lambda$, $\mathbb{E}|Y_1|^s < \infty$.

Proof. If $s < \min\{1, \lambda\}$, then convexity and independence yield:

$$\mathbb{E}|Y_1|^s \leq \sum_{k=0}^{\infty} \mathbb{E}|a_0, a_{-1}, \dots, a_{1-k}|^s \mathbb{E}|b_{-k}|^s,$$

and if $1 \leq s < \lambda$, Hölder inequality yields:

$$(\mathbb{E}|Y_1|^s)^{1/s} \leq \sum_{k=0}^{\infty} (\mathbb{E}|a_0, a_{-1}, \dots, a_{1-k}|^s)^{1/s} (\mathbb{E}|b_{-k}|^s)^{1/s}.$$

But we have $\mathbb{E}|b_{-k}|^s \leq (\mathbb{E}|b_0|^{\lambda+\delta})^{s/(\lambda+\delta)} < \infty$, with δ given by Theorem 1. Besides, the series $\sum_k (\mathbb{E}|a_0, a_{-1}, \dots, a_{1-k}|^s)^{1/s}$ converges thanks to Proposition 1 and Corollary 2. Hence $\mathbb{E}|Y_1|^s < \infty$. \square

Proposition 4. For all i and x , the mappings $t \mapsto G_i(x, t)$ are directly Riemann integrable on \mathbb{R} .

Proof. As G_i are clearly continuous in t , it is sufficient to prove that

$$\sum_{l=-\infty}^{\infty} \sup_{l \leq t < l+1} |G_i(x, t)| < \infty,$$

(see [8]). For all i, x, t , we have $G_i(x, t) = G_i^1(x, t) - G_i^2(x, t)$, where

$$G_i^1(x, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(u - xb_0 < xY_0 a_0 \leq u, a_0 = e_i) du \geq 0,$$

$$G_i^2(x, t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(u < xY_0 a_0 \leq u - xb_0, a_0 = e_i) du \geq 0.$$

For all real t , we have $G_i(x, t) \leq G_i^1(x, t) \leq e^{-t} \int_0^{e^t} u^\lambda du = e^{t\lambda}(\lambda + 1)^{-1}$. In particular, G_i is directly Riemann integrable on \mathbb{R}_- . We still have to study G_i^1 and G_i^2 on \mathbb{R}_+ . These two functions being of the same kind, we only study G_i^1 here.

The rest of the proof is adapted from [16]. Set $\varepsilon \in]0, 1[$ so that $-1 < \lambda - (\lambda + \delta)\varepsilon < 0$, with $\delta > 0$ given by Theorem 1. We have

$$0 \leq e^t G_i^1(x, t) \leq \int_0^{e^t} u^\lambda \mathbb{P}(xb_0 > u^\varepsilon, a_0 = e_i) du$$

$$+ \int_0^{e^t} u^\lambda \mathbb{P}(u - u^\varepsilon < xY_0 a_0 \leq u, a_0 = e_i) du. \quad (9)$$

We are going to give an upper bound for each one of these two terms.

• *First term:*

As $\mathbb{P}(xb_0 > u^\varepsilon, a_0 = e_i) \leq \mathbb{P}(xb_0 > u^\varepsilon)$ we have, as in [16]

$$\int_0^{e^t} u^\lambda \mathbb{P}(xb_0 > u^\varepsilon, a_0 = e_i) du \leq \mathbb{E}|b_0|^{\lambda+\delta} \frac{e^{t(1+\lambda-\varepsilon(\lambda+\delta))}}{1 + \lambda - \varepsilon(\lambda + \delta)}. \quad (10)$$

• *Second term:*

For all $u > 0$ we have $\mathbb{P}(xY_0a_0 > u, a_0 = e_i) \leq \frac{\mathbb{E}|Y_0e_i|^s}{u^s}$ which is finite by Proposition 3. With this slight change in [16], we get

$$\int_0^{e^t} u^\lambda \mathbb{P}(u - u^\varepsilon < xa_0 Y_0 \leq u, a_0 = e_i) du \leq Ce^{t(\lambda + \varepsilon - s)}, \quad (11)$$

where C is a positive constant, and $s \in]0, \lambda[$ is chosen such that $-1 < \lambda + \varepsilon - 1 - s < 0$.

Now let $\alpha = \max\{\lambda + \varepsilon - s; 1 + \lambda - (\lambda + \delta)\varepsilon\} \in]0, 1[$. Eqs. (9)–(11) yield $e^t G_i^1(x, t) \leq Ce^{t\alpha}$ for all positive t , C being another positive constant. Thus $G_i^1(x, t) \leq Ce^{t(\alpha-1)}$ is directly Riemann integrable on \mathbb{R}_+ . \square

4.4. Tail of the stationary distribution

We have proved that F and G satisfy the conditions of Theorem A. Hence for all i, x, t , we have, with the notation of this theorem,

$$Z_i(x, t) \xrightarrow{t \rightarrow \infty} cm_i \sum_{j=1}^p \left[u_j \int_{-\infty}^{\infty} G_j(x, y) dy \right]. \quad (12)$$

Summing up these terms, we get

$$z(x, t) \xrightarrow{t \rightarrow \infty} c \sum_{j=1}^p \left[u_j \int_{-\infty}^{\infty} G_j(x, y) dy \right], \quad (13)$$

as $\sum m_i = 1$. This limit is also the limit of $t^\lambda \mathbb{P}(xY_1 > t)$ by Lemma 9.3 of [9] which is valid under our assumptions (see also Lemma 3.7 of [16] for a similar result). Now it remains to prove that the sum of the two limits for $x \in \{-1, 1\}$ is non-zero.

5. Part II of the proof of Theorem 1

5.1. Special case: b_0 has a constant sign

In Section 3, we have defined the functions

$$\psi_i(x, t) = \mathbb{P}(t - xb_0 < xa_0 Y_0 \leq t, a_0 = e_i) - \mathbb{P}(t < xa_0 Y_0 \leq t - xb_0, a_0 = e_i).$$

If $b_0 \geq 0$ a.s. and $x = 1$, or $b_0 \leq 0$ a.s. and $x = -1$, we have $xb_0 \geq 0$ a.s. and for all i and t ,

$$\begin{aligned} G_i(x, t) &= e^{-t} \int_0^{e^t} u^\lambda \psi_i(x, u) du \\ &= e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(u - xb_0 < xa_0 Y_0 \leq u, a_0 = e_i) du \geq 0. \end{aligned}$$

Similarly, if $xb_0 \leq 0$ a.s. we have for all i and t :

$$G_i(x, t) = -e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(u < xa_0 Y_0 \leq u - xb_0, a_0 = e_i) du \leq 0.$$

Thus, if b_0 has constant sign, for fixed x all $G_i(x, \cdot)$ have constant sign, and have the same sign. Now assume that $\lim z(x, t) = 0$. Then Eq. (13) yields

$$c \sum_{j=1}^p \left[u_j \int_{-\infty}^{\infty} G_j(x, y) dy \right] = 0.$$

As c and all u_j are positive, this yields $G_j(x, t) = 0$ for all j and $t \in \mathbb{R}$. Thus, $Z(x, t) = U * G(x, t) = 0$ for all t , and $z(x, t) = 0$. Hence $\mathbb{P}(xY_1 > t) = 0$ a.s.

- If $b_0 \geq 0$, we have $Y_1 \geq 0$, which contradicts the statement above if $x = 1$. Thus $\lim z(1, t) > 0$. And obviously $\lim z(-1, t) = 0$.
- If $b_0 \leq 0$, we have $Y_1 \leq 0$, which contradicts the statement above if $x = -1$. Thus $\lim z(-1, t) > 0$. And obviously $\lim z(1, t) = 0$.

5.2. Lower bound for $\mathbb{P}(|Y_1| > t)$

Now we study the general case where b_0 is allowed to change sign. We want to prove that there is a positive constant C such that $t^\lambda \mathbb{P}(|Y_1| > t) \geq C > 0$ when t tends to infinity. In the author's opinion, this lower bound is far from obvious. Here we adapt a method proposed by Goldie [9].

Proposition 5. *There is a positive ε and a corresponding positive constant C such that for all large enough t , we have*

$$\mathbb{P}(|Y_1| > t) \geq C \mathbb{P} \left(\sup_n |a_0, \dots, a_{1-n}| > \frac{2t}{\varepsilon} \right).$$

As explained by Goldie [9] for the i.i.d. case, the key for such a lower bound is an inequality established by Grincevičius [10] corresponding to an extension of Lévy's symmetrization inequality: see [6]. We first extend Grincevičius' inequality to the Markovian case.

Recall that $Y_1 = \sum_{k=0}^{\infty} a_0, \dots, a_{1-k} b_{-k}$ and set for $n \geq 1$,

$$Y_1^n = \sum_{k=0}^{n-1} a_0, \dots, a_{1-k} b_{-k} \quad \text{and} \quad \Pi_n = a_0, \dots, a_{1-n}.$$

Let \mathcal{F}_j be the σ -field generated by $(a_{-j}, a_{-j-1}, \dots)$, and X a \mathcal{F}_j -measurable random variable. Let $\text{med}_i(X)$ be a median of X conditionally to $a_{-j} = e_i$, so that $\mathbb{P}(\text{med}_i(X) \leq X | a_{-j} = e_i) \geq \frac{1}{2}$, and $\mathbb{P}(\text{med}_i(X) \geq X | a_{-j} = e_i) \geq \frac{1}{2}$. Set also $\text{med}_-(X) = \min_{1 \leq i \leq p} \{\text{med}_i(X)\}$.

Lemma 2. For all $t > 0$ and $n \geq 1$, we have

$$\mathbb{P}\left(\max_{1 \leq j \leq n} \left\{ Y_1^j + \Pi_j \operatorname{med}_-\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right) \right\} > t\right) \leq 2\mathbb{P}(Y_1^n > t).$$

Proof. Set $T = \inf\{j \leq n \text{ s.t. } Y_1^j + \Pi_j \operatorname{med}_-((Y_1^n - Y_1^j)\Pi_j^{-1}) > t\}$ if this set is not empty, $n + 1$ otherwise, and $B_j = \{\operatorname{med}_-((Y_1^n - Y_1^j)\Pi_j^{-1}) \leq (Y_1^n - Y_1^j)\Pi_j^{-1}\}$. The event $(T = j)$ is in the σ -field generated by $a_0, \dots, a_{1-j}, b_0, \dots, b_{1-j}$, and B_j is in the σ -field generated by $a_{-j}, \dots, a_{1-n}, b_{-j}, \dots, b_{1-n}$. Therefore they are independent conditionally to a_{-j} . Moreover, for all i and j we have

$$\mathbb{P}(B_j | a_{-j} = e_i) \geq \mathbb{P}\left(\operatorname{med}_i\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right) \leq \frac{Y_1^n - Y_1^j}{\Pi_j}\right) \geq \frac{1}{2}.$$

Thus, as Π_j is positive, we have

$$\begin{aligned} \mathbb{P}(Y_1^n > t) &\geq \mathbb{P}\left(\bigcup_{j=1}^n [B_j \cap (T = j)]\right) \\ &= \sum_{j=1}^n \sum_{i=1}^p \mathbb{P}(B_j | a_{-j} = e_i) \mathbb{P}(T = j | a_{-j} = e_i) v(e_i) \\ &\geq \frac{1}{2} \mathbb{P}(T \leq n) \\ &= \frac{1}{2} \mathbb{P}\left(\max_{1 \leq j \leq n} \left\{ Y_1^j + \Pi_j \operatorname{med}_-\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right) \right\} > t\right). \quad \square \end{aligned}$$

Under our assumptions, when n tends to infinity, Y_1^n tends to Y_1 , and for fixed j , $\Pi_j^{-1}(Y_1^n - Y_1^j)$ tends to a random variable \hat{Y} that has the same distribution as Y_1 . Set $m_0 = \operatorname{med}_-(Y_1) = \operatorname{med}_-(\hat{Y})$, and letting n tend to infinity in Lemma 2, yields, for all $t > 0$,

$$\mathbb{P}(\sup_j \{Y_1^j + \Pi_j m_0\} > t) \leq 2\mathbb{P}(Y_1 > t).$$

Replacing Y_1 by $-Y_1$ yields a similar formula for all $t < 0$, hence for all $t > 0$, we have

$$\mathbb{P}(\sup_j |Y_1^j + \Pi_j m_0| > t) \leq 2\mathbb{P}(|Y_1| > t). \quad (14)$$

Furthermore, as proved in Goldie [9, p. 157], we have for all $t > |m_0|$,

$$\mathbb{P}(\sup_n \{Y_1^n + \Pi_n m_0\} > t) \geq \mathbb{P}(\exists n \text{ s.t. } |(Y_1^{n+1} + \Pi_{n+1} m_0) - (Y_1^n + \Pi_n m_0)| > 2t),$$

where $Y_1^0 = 0$ and $\Pi_0 = 1$ by convention. Now notice that:

$$\begin{aligned} (Y_1^{n+1} + \Pi_{n+1} m_0) - (Y_1^n + \Pi_n m_0) &= a_0, \dots, a_{1-n} b_{-n} + (\Pi_{n+1} - \Pi_n) m_0 \\ &= \Pi_n (b_{-n} + (a_{-n} - 1) m_0). \end{aligned}$$

Thus Eq. (14) yields, for all $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P}(|Y_1| > t) &\geq \frac{1}{2} \mathbb{P}(\exists n \text{ s.t. } |\Pi_n(b_{-n} + (a_{-n} - 1)m_0)| > 2t) \\ &\geq \frac{1}{2} \mathbb{P}\left(\exists n \text{ s.t. } |\Pi_n| > \frac{2t}{\varepsilon} \text{ and } |b_{-n} + (a_{-n} - 1)m_0| > \varepsilon\right). \end{aligned} \quad (15)$$

Now we extend Feller–Chung inequality (see [6]).

Lemma 3. *We have, for all $t > |m_0|$ and $\varepsilon > 0$*

$$\begin{aligned} &\mathbb{P}\left(\exists n \text{ s.t. } |\Pi_n| > \frac{2t}{\varepsilon} \text{ and } |b_{-n} + (a_{-n} - 1)m_0| > \varepsilon\right) \\ &\geq \min_{1 \leq i \leq p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon) \mathbb{P}\left(\exists n \text{ s.t. } |\Pi_n| > \frac{2t}{\varepsilon}\right). \end{aligned}$$

Proof. Set $A_0 = \emptyset$, $A_n = \{|\Pi_n| > \frac{2t}{\varepsilon}\}$ and $B_n = \{|b_{-n} + (a_{-n} - 1)m_0| > \varepsilon\}$. Conditionally to a_{-n} , B_n is independent from A_0, \dots, A_n . Therefore, we have

$$\begin{aligned} \mathbb{P}\left(\bigcup_{n=1}^{\infty} [A_n \cap B_n]\right) &= \sum_{n=1}^{\infty} \mathbb{P}\left(B_n \cap A_n \bigcap_{j=0}^{n-1} [B_j \cap A_j]^c\right) \\ &\geq \sum_{n=1}^{\infty} \mathbb{P}\left(B_n \cap A_n \bigcap_{j=0}^{n-1} A_j^c\right) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^p \mathbb{P}(B_n | a_{-n} = e_i) \mathbb{P}\left(A_n \bigcap_{j=0}^{n-1} A_j^c | a_{-n} = e_i\right) v(e_i). \end{aligned}$$

where A^c denotes the complementary set of A . But the stationarity of (a_n, b_n) , and the independence of these two sequences yield $\mathbb{P}(B_n | a_{-n} = e_i) = \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)$. Thus, we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} [A_n \cap B_n]\right) \geq \min_{1 \leq i \leq p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon) \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right). \quad \square$$

Proof of Proposition 5. Eq. (15) and Lemma 3 yield, for all $t > |m_0|$ and for all $\varepsilon > 0$,

$$\mathbb{P}(|Y_1| > t) \geq \frac{1}{2} \min_{1 \leq i \leq p} \mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon) \mathbb{P}\left(\exists n \text{ s.t. } |\Pi_{n-1}| > \frac{2t}{\varepsilon}\right).$$

If b_0 is not constant (otherwise we get a special case studied in Section 5.1), we can find a $\varepsilon > 0$ such that $\min_{1 \leq i \leq p} \{\mathbb{P}(|b_0 + (e_i - 1)m_0| > \varepsilon)\} > 0$. Thus, as expected,

there is a positive constant C such that for all $t > |m_0|$, we have

$$\mathbb{P}(|Y_1| > t) \geq C \mathbb{P}\left(\sup_n |\Pi_n| > \frac{2t}{\varepsilon}\right). \quad \square$$

5.3. Study of the product a_0, \dots, a_{1-n}

To estimate the probability $\mathbb{P}(\sup_n |\Pi_n| > t)$, we use the method of Arjas and Speed [1], and Renewal Theorem B. First, we introduce some notation. Let $S_0 = 0$ and for all positive n ,

$$S_n = \sum_{k=1}^n \log(a_{1-k}) = \log(a_0, \dots, a_{1-n}) = \log \Pi_n.$$

The process (a_{1-n}, S_n) is called a Markov-modulated random walk: see [3,2], or a Markov renewal process: see [1], with semi-Markov matrix $Q = (q_{ij})$, where:

$$q_{ij}(t) = \mathbb{P}(a_{-n} = e_j, \log a_{-n} \leq t \mid a_{1-n} = e_i) = \mathbf{1}_{t \geq \log e_j} \frac{v(e_j)}{v(e_i)} p_{ji}.$$

The *first ladder epoch* of the random walk (S_n) is $\tau = \tau_1 = \inf\{n \geq 1 \text{ s.t. } S_n > 0\}$, and the *first ladder height* is S_τ . Let $H(t)$ be the semi-Markov matrix of this ladder process:

$$H_{ij}(t) = \mathbb{P}(\tau < \infty, S_\tau \leq t, a_{1-\tau} = e_j \mid a_1 = e_i).$$

As $S_\tau > 0$, H is distributed on the positive half-line.

We have $S_{\tau-1} \leq 0$ and $S_\tau > 0$, which implies that $\log(a_{1-\tau}) > 0$, i.e. $a_{1-\tau} > 1$. Let us rearrange the e_i such that $e_1, \dots, e_q > 1$ and $e_{q+1}, \dots, e_p \leq 1$ (they cannot be all smaller than or equal to one, for otherwise P'_α would be a sub-stochastic matrix for all α which is impossible as $\rho(P'_\alpha) > 1$ for all $\alpha > \lambda$ thanks to the convexity property). Thus, for all $j > q$, we have $H_{ij}(t) = 0$ for all t . Let \overline{H} be the sub-matrix $(H_{ij})_{1 \leq i, j \leq q}$. Besides, S_τ cannot be greater than $\max_i \log(e_i)$, thus H (and \overline{H}) have finite support.

We define also the n th ladder epoch by $\tau_n = \inf\{k > \tau_{n-1} \text{ s.t. } S_k > S_{\tau_{n-1}}\}$, and S_{τ_n} is the corresponding ladder height. We check that

$$H_{ij}^{(n)}(t) = \mathbb{P}(\tau_n < \infty, S_{\tau_n} \leq t, a_{1-\tau_n} = e_j \mid a_1 = e_i),$$

where $H^{(n)}$ is the n -fold convolution of H . We also have $\overline{H^{(n)}} = \overline{H}^{(n)}$, with obvious notation. Let $\Psi = \sum_{n=0}^{\infty} H^{(n)}$ be the renewal function associated with H and $\overline{\Psi}$ the one associated with \overline{H} . Finally, let $M = \sup_n S_n = \sup_n S_{\tau_n}$ be the maximum of our random walk. We have, for all $1 \leq i \leq p$:

$$\mathbb{P}(M \leq t \mid a_1 = e_i) = \sum_{j=1}^p \left[\Psi_{ij}(t) \left(1 - \sum_{k=1}^p H_{jk}(\infty) \right) \right],$$

and if $i \leq q$ it reduces to

$$\mathbb{P}(M \leq t \mid a_1 = e_i) = \sum_{j=1}^q \left[\bar{\Psi}_{ij}(t) \left(1 - \sum_{k=1}^q \bar{H}_{jk}(\infty) \right) \right]. \quad (16)$$

Now, we are going to apply renewal Theorem B, with $F = \bar{H}$ and $\alpha = \lambda$ (here it is easier to apply Theorem B than to check the four assumptions of Arjas and Speed [1]). As $H(0) = (0)$, we have $\rho(\bar{H}(0)) < 1$, and as all H_{ij} are probabilities, $\bar{H}(\infty)$ is finite. In addition, \bar{B} , the expectation of $\bar{H}_\lambda(\infty) = \int_0^\infty e^{-\lambda u} \bar{H}(du)$ is finite as \bar{H} has finite support. The assumption that the $\log e_i$ are not integral multiples of the same number also implies that \bar{H} is non-lattice.

We have

$$\begin{aligned} \bar{H}_{ij}(\infty) &= \mathbb{P}(\tau < \infty, a_{1-\tau} = e_j \mid a_1 = e_i) \\ &\geq \mathbb{P}(\tau = 1, a_{1-\tau} = e_j \mid a_1 = e_i) \\ &= \mathbb{P}(a_0 = e_j \mid a_1 = e_i) = p_{ji} \frac{v(e_j)}{v(e_i)}. \end{aligned}$$

As all $v(e_i)$ are positive, and P is irreducible and aperiodic, this implies that $\bar{H}(\infty)$ also is irreducible and aperiodic.

Note that $H(\infty)$ and $\bar{H}(\infty)$ have the same spectral radius. Indeed, $H(\infty)$ is a block-triangular matrix with first diagonal block $\bar{H}(\infty)$ and second diagonal block (0) . Therefore $\rho(H(\infty)) = \rho(\bar{H}(\infty))$.

To compute the spectral radius of $\bar{H}_\lambda(\infty)$, we introduce $\hat{Q}(s) = (\hat{q}_{ij}(s))$, the moment generating function of Q , as in [1]:

$$\hat{q}_{ij}(s) = \int e^{st} q_{ij}(dt) = e_j^s \frac{v(e_j)}{v(e_i)} p_{ji} = \Delta^{-1} P_s \Delta,$$

where $\Delta = \text{diag}(e_i^s v(e_i))$. Thus P_s and $\hat{Q}(s)$ have the same spectral radius, and in particular $\rho(\hat{Q}(\lambda)) = 1$. In addition, $\hat{Q}(\lambda)$ is a non-negative irreducible matrix, as P_λ is, therefore, by Perron–Frobenius Theorem it possesses a right eigenvector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_p)'$ with positive coordinates. Set $E = \text{diag}(\varepsilon_i)$. Then

$$Q_\lambda(t) = E^{-1} \left(\int_{-\infty}^t e^{\lambda u} Q(du) \right) E$$

is a semi-Markov matrix, and let $(\lambda a_{1-n}, \lambda S_n)$ be its associated Markov renewal process. As proved in [1], $EH_\lambda(\infty)E^{-1}$ is the semi-Markov matrix of the ascending ladder process of (λS_n) , and the mean of $\log \lambda a_{1-n}$ is the derivative of $s \mapsto \log \rho(P_s)$ at λ . But we have $\log \rho(P_0) = \log \rho(P_\lambda) = 0$, its right-hand derivative at zero is negative (Proposition 2) and this function is convex (Corollary 1). Thus its derivative at λ is positive, and λS_n drifts to $+\infty$. Proposition 4.2 of [2] then implies that $\rho(EH_\lambda(\infty)E^{-1}) = 1 = \rho(\bar{H}_\lambda(\infty))$.

We have proved that all assumptions of Theorem B are valid. Thus Eq. (16) yields, when t tends to infinity

$$\begin{aligned} 1 - \mathbb{P}(M \leq t) &= \sum_{j=1}^q \left(1 - \sum_{k=1}^q \overline{H}_{jk}(\infty) \right) \int_t^\infty e^{-\lambda u} (e^{\lambda u} \overline{\Psi}_{ij})(du) \\ &\sim \sum_{j=1}^q \left(1 - \sum_{k=1}^q \overline{H}_{jk}(\infty) \right) \int_t^\infty e^{-\lambda u} \overline{c m}_i \overline{u}_j du \\ &= \sum_{j=1}^q \left(1 - \sum_{k=1}^q \overline{H}_{jk}(\infty) \right) \frac{\overline{c m}_i \overline{u}_j}{\lambda} e^{-\lambda t}, \end{aligned} \quad (17)$$

where \overline{m} and \overline{u} are right and left positive eigenvectors of $\overline{H}_\lambda(\infty)$, with the same normalization as in Section 2, and $\overline{c} = (\overline{u} \overline{B} \overline{m})^{-1} > 0$. Proposition 4.2 of [2] implies that $\rho(\overline{H}(\infty)) = \rho(H(\infty)) < 1$ as $\mathbb{E} \log |a_0| < 0$ (Assumption (2)). Therefore $\overline{H}(\infty)$ is strictly sub-stochastic and there is a $j \leq q$ such that $1 - \sum_{k=1}^q \overline{H}_{jk}(\infty) > 0$. Hence the right-hand side term of Eq. (17) is positive, thus we have, when t tends to infinity,

$$e^{\lambda t} \mathbb{P}(M > t) \geq \sum_{i=1}^q e^{\lambda t} \mathbb{P}(M > t | a_1 = e_i) v(e_i) \geq C > 0. \quad (18)$$

Now Eq. (18) and Proposition 5 yield, for large enough t :

$$t^\lambda \mathbb{P}(|Y_1| > t) \geq C > 0$$

and thus with the notation of Theorem 1 we have $L(-1) + L(1) > 0$.

6. Proof of Theorem 2

Assume that the hypotheses of Theorem 2 are satisfied. Our aim is to apply Theorem A to the distribution matrix \tilde{F} and the vector \tilde{G} defined in Section 3.2. As in the positive case, notice that $\tilde{F}_{ij}(\infty) < \infty$ and that the expectation \tilde{B} of \tilde{F} is well defined. The assumption that the $\log |e_i|$ are not integral multiples of the same number implies again that \tilde{F} is non-lattice.

For the other points, we use the previous results obtained in the positive case. For all real t , set $F(t) = (|e_i|^\lambda p_{ji} \mathbf{1}_{t \geq \log |e_i|})_{1 \leq i, j \leq p}$. It is non-negative, and

$$\tilde{F} = \begin{pmatrix} (F)_{1 \leq i \leq \ell, 1 \leq j \leq p} & (0) \\ (0) & (F)_{\ell+1 \leq i \leq p, 1 \leq j \leq p} \\ (0) & (F)_{1 \leq i \leq \ell, 1 \leq j \leq p} \\ (F)_{\ell+1 \leq i \leq p, 1 \leq j \leq p} & (0) \end{pmatrix}. \quad (19)$$

6.1. Irreducibility

We have seen in the positive case that $F(\infty)$ is irreducible. Unfortunately, this does not always imply that $\tilde{F}(\infty)$ is also irreducible.

Definition 3. Let $A = (a_{ij})_{1 \leq i, j \leq p}$ be a positive matrix, and $0 \leq \ell \leq p - 1$ an integer. A is ℓ -reducible if there is (I, J) a (possibly trivial) partition of $\{1, \dots, p\}$ such that

For all $1 \leq i \leq \ell$

if $i \in I$, then $a_{ij} = 0 \ \forall j \in J$,

if $i \in J$, then $a_{ij} = 0 \ \forall j \in I$.

For all $\ell + 1 \leq i \leq p$

if $i \in I$, then $a_{ij} = 0 \ \forall j \in I$,

if $i \in J$, then $a_{ij} = 0 \ \forall j \in J$.

If A is not ℓ -reducible, we say that A is ℓ -irreducible.

We gave this definition in order to have the following proposition.

Proposition 6. Let $A = (a_{ij})_{1 \leq i, j \leq p}$ be a positive irreducible matrix, and $0 \leq \ell \leq p - 1$ an integer. Then, the matrix B defined as follows:

$$B = \begin{pmatrix} (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq p} & (0) \\ (0) & (a_{ij})_{\ell+1 \leq i \leq p, 1 \leq j \leq p} \\ (0) & (a_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq p} \\ (a_{ij})_{\ell+1 \leq i \leq p, 1 \leq j \leq p} & (0) \end{pmatrix}$$

is irreducible if and only if A is ℓ -irreducible.

Proof. Suppose A is ℓ -reducible for a partition (I, J) . Set $\tilde{I} = I \cup (J + p)$ and $\tilde{J} = J \cup (I + p)$, so that (\tilde{I}, \tilde{J}) is a non-trivial partition of $\{1, \dots, 2p\}$. Then for all $(i, j) \in \tilde{I} \times \tilde{J}$ we can prove that $b_{ij} = 0$ and $b_{ji} = 0$. Thus B is reducible.

Suppose that B is reducible for the non trivial partition (I, J) . Set:

$$I_1 = I \cap \{1, \dots, p\}, \quad I_2 = I \cap \{p + 1, \dots, 2p\},$$

$$J_1 = J \cap \{1, \dots, p\}, \quad J_2 = J \cap \{p + 1, \dots, 2p\}.$$

We can prove that $I_1 = J_2 - p$ and $I_2 = J_1 + p$, and we check that A is ℓ -reducible for the partition (I_1, J_1) . \square

Now we distinguish two cases according to whether P' is ℓ -reducible or not.

6.2. First case: P' is ℓ -irreducible

In this case $F(\infty)$ is also ℓ -irreducible for λ given by Theorem 2 and $\tilde{F}(\infty)$ is irreducible. In addition, we have $\|\tilde{F}(\infty)^n\| \leq \|F(\infty)^n\|$ for all n . As $F(\infty)$ is aperiodic, this sequence is bounded. We know that $F(\infty)$ has spectral radius 1. The same also holds for $\tilde{F}(\infty)$ thanks to the following lemma:

Lemma 4. *If the matrix $A = (a_{ij})_{i \leq i, j \leq p}$ is non-negative, then the matrix B of Proposition 6 has the same spectral radius as A .*

Proof. Let us compute $\mathcal{P}(X) = \det(B - XI_{2p})$ the characteristic polynomial of B . Adding the last p columns of $B - XI_{2p}$ to the first p columns, then subtracting the first p rows to the last p rows, we get $\det(B - XI_{2p}) = \det(A - XI_p) \det(A_1 - XI_p)$, where A_1 is the following matrix:

$$A_1 = \begin{pmatrix} (a_{ij})_{1 \leq i \leq l, 1 \leq j \leq p} \\ (-a_{ij})_{l+1 \leq i \leq p, 1 \leq j \leq p} \end{pmatrix}.$$

Thus the spectral radius of B is the maximum of that of A and that of A_1 . But A is non-negative, and component-wise $|A_1| = A$, so Theorem 8.1.18 of [12] yields $\rho(A_1) \leq \rho(A)$. Thus $\rho(B) = \rho(A)$. \square

Note that if λ is an eigenvalue of A with eigenvector X , then we have $B(X', X')' = ((AX)', (AX'))' = \lambda(X', X')'$, thus $(X', X')'$ is an eigenvector of B for the same eigenvalue. Let m and u be positive right and left eigenvectors of F for the eigenvalue 1, so that $\sum m_i = \sum m_i u_i = 1$. Then $\tilde{m} = 2^{-1}(m', m')'$ is a right eigenvector of \tilde{F} for the eigenvalue 1, and satisfies $\sum \tilde{m}_i = 1$. And $\tilde{u} = (u', u')'$ is a left eigenvector so that $\sum_{i=1}^{2p} \tilde{u}_i \tilde{m}_i = \sum_{i=1}^p u_i m_i = 1$.

6.2.1. Properties of \tilde{F} and \tilde{G}

Let $\tilde{U} = \sum_{k=0}^{\infty} \tilde{F}^{(k)}$. As $\tilde{F}_{ij} \leq F_{ij}$, the same holds for their k -fold convolution. Set $U = \sum_{k=0}^{\infty} F^{(k)}$, then $U(t) < \infty$ as in the positive case, and thus $\tilde{U}(t) < \infty$.

To prove that $\tilde{Z} = \tilde{U} * \tilde{G}$, it is sufficient to prove that $\tilde{F}^{(n)} * \tilde{Z} \xrightarrow{n \rightarrow \infty} 0$. But we have seen in Section 3 that

$$\begin{aligned} (\tilde{F} * \tilde{Z})_i(t) &= \sum_{j=1}^p e^{-(t - \log |e_i|)} \int_0^{e^{t - \log |e_i|}} |e_i|^\lambda p_{ji} u^\lambda \mathbb{P}(\pm Y_1 > u, a_0 = e_j) du \\ &= e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(\pm a_0 Y_0 > u, a_0 = e_i) du. \end{aligned}$$

Similarly, we get

$$(\tilde{F}^{(n)} * \tilde{Z})_i(t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(\pm a_{1-n}, \dots, a_0 Y_{1-n} > u, a_0 = e_i) du.$$

And thus, as in the positive case, we have

$$\sum_{i=1}^p (\tilde{F}^{(n)} * \tilde{Z})_i(t) = e^{-t} \int_0^{e^t} u^\lambda \mathbb{P}(\pm a_{1-n}, \dots, a_0 Y_{1-n} > u) du.$$

But Eq. (8) implies $a_{1-n}, \dots, a_0 \rightarrow 0$. Thus for all $u > 0$, the bounded convergence theorem yields $\mathbb{P}(\pm a_{1-n}, \dots, a_0 Y_{1-n} > u) \rightarrow 0$, because $Y < \infty$ a.s. and is stationary.

Thus $\sum_{i=1}^p (\tilde{F}^{(n)} * \tilde{Z})_i(t) \rightarrow 0$, and as all the terms in the sum are non-negative, each one tends to 0 and we have, as expected $\tilde{Z} = \tilde{U} * \tilde{G}$.

We have $\tilde{G}_i(t) = G_i(\pm 1, t)$ which is directly Riemann integrable under the assumptions of Theorem 2 as seen for the positive case.

6.2.2. Tail of the distribution

We have proved that \tilde{F} and \tilde{G} satisfy the assumptions of Theorem A. Hence for all i, t , we have, with obvious notations,

$$\tilde{Z}_i(t) \xrightarrow{t \rightarrow \infty} \tilde{c} \tilde{m}_i \sum_{j=1}^{2p} \left[\tilde{u}_j \int_{-\infty}^{\infty} \tilde{G}_j(y) dy \right]. \quad (20)$$

Notice that $\tilde{c} = c$. Indeed, we have

$$\tilde{u}' \tilde{B} \tilde{m} = \frac{1}{2} (u', u') \begin{pmatrix} (b_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq p} & 0 \\ 0 & (b_{ij})_{\ell+1 \leq i \leq p, 1 \leq j \leq p} \\ 0 & (b_{ij})_{1 \leq i \leq \ell, 1 \leq j \leq p} \\ (b_{ij})_{\ell+1 \leq i \leq p, 1 \leq j \leq p} & 0 \end{pmatrix} \begin{pmatrix} m \\ m \end{pmatrix}.$$

Hence $\tilde{c}^{-1} = \frac{1}{2} (u' B m + u' B m) = c^{-1}$, where B is the expectation of F . Thus summing up the term in Eq. (20), we get

$$z(x, t) \xrightarrow{t \rightarrow \infty} c \sum_{j=1}^p \left[u_j \int_{-\infty}^{\infty} (G_j(-1, y) + G_j(1, y)) dy \right].$$

And we use again Lemma 9.3 of [9] to conclude that $t^\ell \mathbb{P}(x Y_1 > t)$ has the same limit. Note that here this limit does not depend on x , therefore both $t^\ell \mathbb{P}(Y_1 > t)$ and $t^\ell \mathbb{P}(Y_1 < -t)$ have the same limit.

6.3. Second case: P' is l -reducible

As seen in the proof of Proposition 6, there is (I, J) a non-trivial partition of $\{1, \dots, 2p\}$ such that for all (i, j) in $I \times J$ we have $\tilde{F}_{ij}(\infty) = \tilde{F}_{ji}(\infty) = 0$. Suppose that 1 belongs to I . Then System (5) splits into two independent systems of size p , one with the components $(\tilde{Z}_i)_{i \in I}$ and the other with $(\tilde{Z}_i)_{i \in J}$. Each of these systems has associated matrix F that satisfies the hypothesis of Renewal Theorem A, as seen in the positive case. For all i , \tilde{G}_i is also directly Riemann integrable as seen in the preceding section. Thus Theorem A yields

$$\forall i \in I, \quad \tilde{Z}_i(t) \xrightarrow{t \rightarrow \infty} c m_{\bar{i}} \sum_{\substack{1 \leq j \leq 2p \\ j \in I}} u_{\bar{j}} \int_{-\infty}^{\infty} \tilde{G}_j(y) dy,$$

$$\forall i \in J, \quad \tilde{Z}_i(t) \xrightarrow{t \rightarrow \infty} c m_{\bar{i}} \sum_{\substack{1 \leq j \leq 2p \\ j \in J}} u_{\bar{j}} \int_{-\infty}^{\infty} \tilde{G}_j(y) dy,$$

where \bar{i} denotes i if $i \leq p$ and $i - p$ if $i > p$. Summing up these equalities, we get

$$z(1, t) \xrightarrow{t \rightarrow \infty} c \sum_{j=1}^p u_j \int_{-\infty}^{\infty} (\mathbf{1}_I(j) G_j(1, y) + \mathbf{1}_J(j) G_j(-1, y)) dy$$

and

$$z(-1, t) \xrightarrow{t \rightarrow \infty} c \sum_{j=1}^p u_j \int_{-\infty}^{\infty} (\mathbf{1}_J(j) G_j(1, y) + \mathbf{1}_I(j) G_j(-1, y)) dy.$$

Again, $t^\lambda \mathbb{P}(x Y_1 > t)$ has the same limit as $z(x, t)$ for $x \in \{-1, 1\}$. Note that here these two limits are possibly different.

6.4. The sum of the limits is non-zero

The proof is the same for both cases. The results of Section 5 can be extended to the present case. The result of Section 5.1 about the special case when b_0 has constant sign is valid here. Thus if both limits are zero then $Y_0 = 0$ almost surely which is impossible as 0 is not a solution of Eq. (1).

If X is a random variable, set $\text{med}_+(X) = \max_{1 \leq i \leq p} \{\text{med}_i(X)\}$. The analogous of Lemma 2 is as follows:

Lemma 5. *For all $t > 0$ and $n \geq 1$, we have*

$$\begin{aligned} 2\mathbb{P}(Y_1 > t) \geq & \mathbb{P}\left(\max_{1 \leq j \leq n} \left\{ \mathbf{1}_{\Pi_j > 0} \left[Y_1^j + \Pi_j \text{med}_- \left(\frac{Y_1^n - Y_1^j}{\Pi_j} \right) \right] \right\} > t\right) \\ & + \mathbb{P}\left(\max_{1 \leq j \leq n} \left\{ \mathbf{1}_{\Pi_j < 0} \left[Y_1^j + \Pi_j \text{med}_+ \left(\frac{Y_1^n - Y_1^j}{\Pi_j} \right) \right] \right\} > t\right) \end{aligned}$$

Proof. As Π_j is not always positive, we introduce new events, depending on the sign of Π_j : set $T_+ = \inf\{j \leq n \text{ s.t. } \Pi_j > 0 \text{ and } Y_1^j + \Pi_j \text{med}_-((Y_1^n - Y_1^j)\Pi_j^{-1}) > t\}$ if this set is not empty, $n + 1$ otherwise, $T_- = \inf\{j \leq n \text{ s.t. } \Pi_j < 0 \text{ and } Y_1^j + \Pi_j \text{med}_+((Y_1^n - Y_1^j)\Pi_j^{-1}) > t\}$ if it is not empty, $n + 1$ otherwise, $B_j^+ = \{\text{med}_-((Y_1^n - Y_1^j)\Pi_j^{-1}) \leq (Y_1^n - Y_1^j)\Pi_j^{-1}\}$, and $B_j^- = \{\text{med}_+((Y_1^n - Y_1^j)\Pi_j^{-1}) \geq (Y_1^n - Y_1^j)\Pi_j^{-1}\}$. The events $(T_+ = j)$ and $(T_- = j)$ on the one hand and B_j^+ and B_j^- on the other hand are independent conditionally to a_{-j} . Moreover, for all i, j we have,

$$\mathbb{P}(B_j^+ | a_{-j} = e_i) \geq \mathbb{P}\left(\text{med}_i\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right) \leq \frac{Y_1^n - Y_1^j}{\Pi_j}\right) \geq \frac{1}{2}$$

and

$$\mathbb{P}(B_j^- | a_{-j} = e_i) \geq \mathbb{P}\left(\text{med}_i\left(\frac{Y_1^n - Y_1^j}{\Pi_j}\right) \geq \frac{Y_1^n - Y_1^j}{\Pi_j}\right) \geq \frac{1}{2}.$$

Thus we get, as in the proof of Lemma 2:

$$\begin{aligned} \mathbb{P}(Y_1^n > t) &\geq \mathbb{P}\left(\bigcup_{j=1}^n [(T_+ = j) \cap B_j^+] \cup [(T_- = j) \cap B_j^-]\right) \\ &\geq \frac{1}{2} (\mathbb{P}(T_+ \leq n) + \mathbb{P}(T_- \leq n)) \\ &= \frac{1}{2} \left[\mathbb{P}\left(\max_{1 \leq j \leq n} \left\{ \mathbf{1}_{\Pi_j > 0} \left[Y_1^j + \Pi_j \operatorname{med}_- \left(\frac{Y_1^n - Y_1^j}{\Pi_j} \right) \right] \right\} > t \right) \right. \\ &\quad \left. + \mathbb{P}\left(\max_{1 \leq j \leq n} \left\{ \mathbf{1}_{\Pi_j < 0} \left[Y_1^j + \Pi_j \operatorname{med}_+ \left(\frac{Y_1^n - Y_1^j}{\Pi_j} \right) \right] \right\} > t \right) \right]. \quad \square \end{aligned}$$

The rest of the proof runs the same as in the positive case for each of these two terms. Set $m_- = \operatorname{med}_-(Y_1)$ and $m_+ = \operatorname{med}_+(Y_1)$. For all $\varepsilon > 0$ and $t > \max\{|m_+|, |m_-|\}$, we get

$$\begin{aligned} &\mathbb{P}\left(\exists n \text{ s.t. } \Pi_n > \frac{2t}{\varepsilon} \text{ and } |b_{-n} + (a_{-n} - 1)m_-| > \varepsilon\right) \\ &\geq \min_{1 \leq i \leq p} \mathbb{P}(|b_0 + (e_i - 1)m_-| > \varepsilon) \mathbb{P}\left(\exists n \text{ s.t. } \Pi_n > \frac{2t}{\varepsilon}\right), \end{aligned}$$

and

$$\begin{aligned} &\mathbb{P}\left(\exists n \text{ s.t. } \Pi_n < -\frac{2t}{\varepsilon} \text{ and } |b_{-n} + (a_{-n} - 1)m_+| > \varepsilon\right) \\ &\geq \min_{1 \leq i \leq p} \mathbb{P}(|b_0 + (e_i - 1)m_+| > \varepsilon) \mathbb{P}\left(\exists n \text{ s.t. } \Pi_n < -\frac{2t}{\varepsilon}\right). \end{aligned}$$

If b_0 is not constant, we can again find $\varepsilon > 0$ such that $\min_{1 \leq i \leq p} \{\mathbb{P}(|b_0 + (e_i - 1)m_-| > \varepsilon)\} > 0$ and $\min_{1 \leq i \leq p} \{\mathbb{P}(|b_0 + (e_i - 1)m_+| > \varepsilon)\} > 0$. Thus, we get the analogous of Proposition 5: there is a constant $C > 0$ and $\varepsilon > 0$ such that for all large enough t :

$$\mathbb{P}(|Y_1| > t) \geq C \mathbb{P}\left(\sup_n |\Pi_n| > \frac{2t}{\varepsilon}\right).$$

Define the new random walk $S_n = \log |a_0, \dots, a_{1-n}|$. With this slight change in Section 5.3, the proof is the same.

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